EAS 8803 - Obs. Seismology
Lec#2: Fourier Transform/Linear System

Dr. Zhigang Peng, Spring 2013

Figure 6.2-3: Amplitude spectra for the body and surface wave segments from a large earthquake.

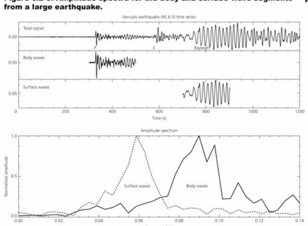
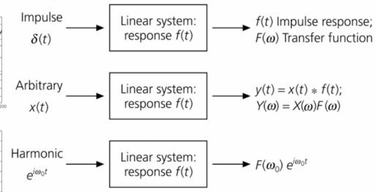


Figure 3.3-29: Seismic section before and after deconvolution.



Last Time

- Course Introduction
  - Class logistics, requirements and policies
  - Class schedule
- Introduction to digital signal processing and its relation to seismological research
- Fourier Series/Fourier Transform

Reading: Stein and Wysession Chap. 6.1 – 6.2

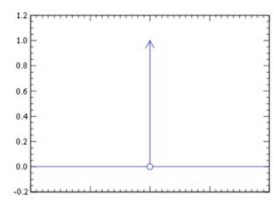
Today's Outline

- Fourier transforms
- Linear systems

Reading: Stein and Wysession Chap. 6.2-6.3

What is a Delta Function?

- The Dirac delta function, or δ function, is (informally) a generalized function depending on a real parameter such that it is zero for all values of the parameter except when the parameter is zero, and its integral over the parameter from $-\infty$ to ∞ is equal to one. (From wipipedia)



Delta function

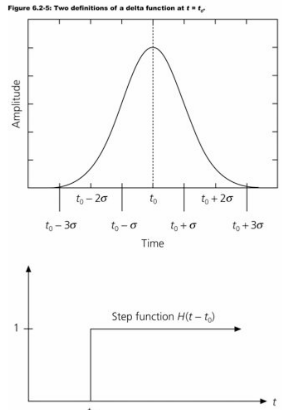
Three ways to define it

$$\delta(t-t_0) = \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]$$

$$f(t_0) = \int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt$$

$$\delta(t-t_0) = dH(t-t_0)/dt$$

Figure 6.2-5: Two definitions of a delta function at $t = t_0$.



Fourier transform of the delta function

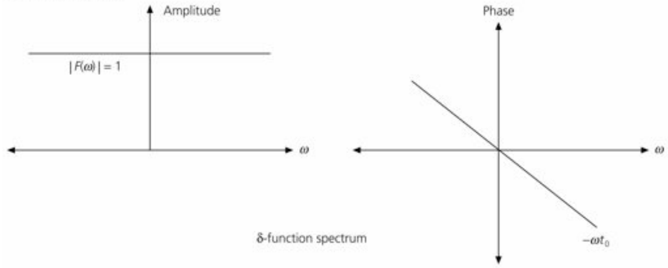
- To find the Fourier transform of the delta function, we use the definition of the transform with $f(t) = \delta(t-t_0)$

$$F(\varpi) = \int_{-\infty}^{\infty} e^{-i\varpi t}\delta(t-t_0)dt = e^{-i\varpi t_0}$$

- The amplitude spectrum is $|F(\varpi)| = \left(e^{-i\varpi t_0} e^{i\varpi t_0}\right)^{1/2} = 1$

- The phase spectrum is $\phi(\varpi) = \varpi t_0$

Figure 6.2-6: Amplitude and phase spectra of the Fourier transform of a delta function.



- If the delta function is at time zero,

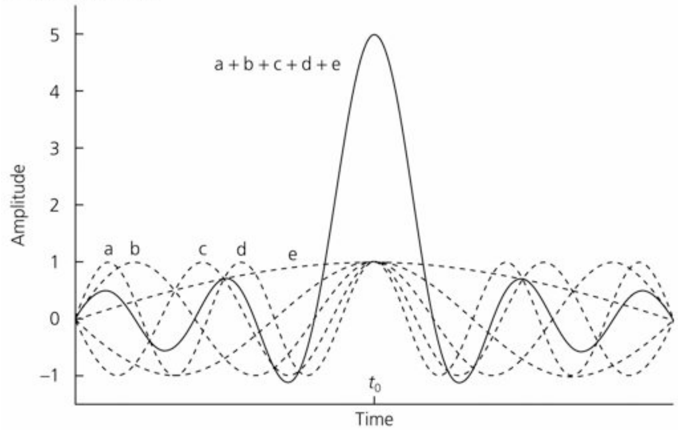
$$F(\varpi) = \int_{-\infty}^{\infty} e^{-i\varpi t}\delta(t)dt = 1$$

Fourier transform of the delta function

- The delta function's amplitude spectrum has unit amplitude at all frequencies.
- The output from a linear time-invariant system with delta function input is called impulse response (in time domain), and transfer function (in frequency domain).
- The inverse transform of the delta function

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\varpi t_0} e^{i\varpi t} d\varpi = \delta(t - t_0)$$

Figure 6.2-7: Fourier transform of a delta function as the sum of sinusoids of all frequencies.



Delta function in the frequency domain

- A delta function at angular frequency ϖ_0 has an inverse transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\varpi - \varpi_0) e^{i\varpi t} d\varpi = \frac{1}{2\pi} e^{i\varpi_0 t}$$

- So we can express the delta function in terms of its Fourier transform

$$\delta(\varpi - \varpi_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\varpi_0 t} e^{-i\varpi t} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\varpi_0 - \varpi)t} dt$$

Delta function in the frequency domain

- Delta function in angular frequency give the spectra of sinusoids with a single frequency.
- For example, a cosine with frequency ϖ_0

$$f(t) = \cos\varpi_0 t = (e^{i\varpi_0 t} + e^{-i\varpi_0 t})/2$$

- Has a Fourier transform

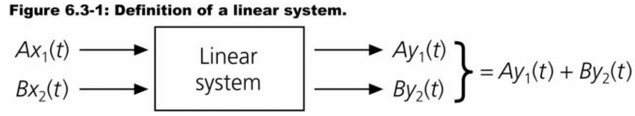
$$F(\varpi) = \frac{1}{2}\int_{-\infty}^{\infty} (e^{i\varpi_0 t} + e^{-i\varpi_0 t}) e^{-i\varpi t} dt = \frac{1}{2}\int_{-\infty}^{\infty} (e^{i(\varpi_0 - \varpi)t} + e^{-i(\varpi_0 + \varpi)t}) dt$$

$$F(\varpi) = \pi[\delta(\varpi - \varpi_0) + \delta(\varpi + \varpi_0)]$$

Linear Systems

- A "**system**" is a general representation of any device or processes that takes an input signal and modifies it.
- A "**linear system**" is is defined by the following diagram, and is previously referred as the principle of *superposition*.

Figure 6.3-1: Definition of a linear system.

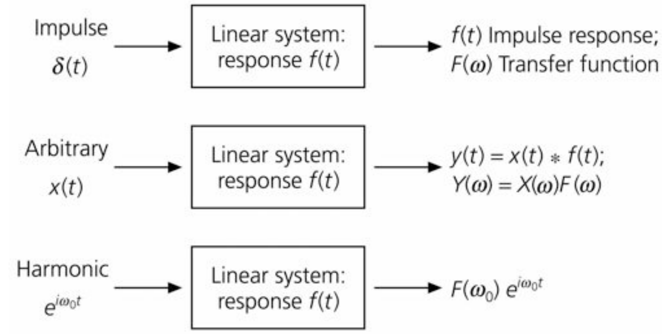


Linear Systems

- The earth generally behaves as a "*linear system*" when transmitting seismic waves.
- Hence, linear system models are used in a wide variety of seismological applications.
- Fourier analysis is a natural tool for studying linear systems because Fourier transform has the same linear properties.
- Can you think of any cases when the Earth is behaving as a "nonlinear system"?

Impulse Response of a Linear System

Figure 3.3-29: Seismic section before and after deconvolution.



Linear Systems

- The output spectrum of an arbitrary input signal

$$Y(\varpi) = X(\varpi)F(\varpi)$$

- The output in the time domain $y(t)$ can be found

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\varpi)F(\varpi)e^{i\varpi t}d\varpi$$

- For the impulse $x(t) = \delta(t)$, $X(\varpi) = 1$, $y(t) = f(t)$
- For a harmonic input signal $x(t) = e^{i\varpi_0 t}$
- The transform is the delta function in frequency domain $X(\varpi) = 2\pi\delta(\varpi - \varpi_0)$. The output is

$$Y(\varpi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\varpi - \varpi_0)F(\varpi)e^{i\varpi t}d\varpi = F(\varpi_0)e^{i\varpi_0 t}$$

Relation between the input/output and the impulse response

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\varpi)F(\varpi)e^{i\varpi t}d\varpi$$

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)e^{-i\varpi\tau}d\tau\right]\left[\int_{-\infty}^{\infty} x(\tau')e^{-i\varpi\tau'}d\tau'\right]e^{i\varpi t}d\varpi$$

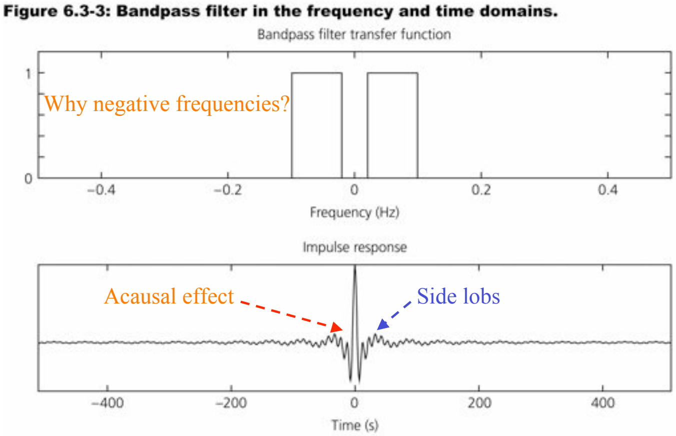
$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)f(\tau')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\varpi(t-\tau'-\tau)}d\varpi\right]d\tau d\tau'$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} f(\tau')\delta(t-\tau'-\tau)d\tau'\right]d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)f(t-\tau)d\tau \dashrightarrow y(t) = x(t) * f(t)$$

Convolution

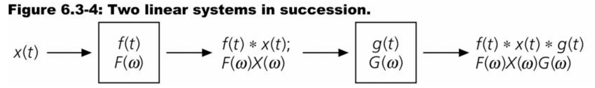
Figure 6.3-3: Bandpass filter in the frequency and time domains.



Convolution and deconvolution modeling in seismology

- Linear system ideas are pervasive in seismology.
- If a signal $x(t)$ goes through two linear systems in succession with impulse response $f(t)$ and $g(t)$, the output is either a convolution in the time domain, or the product of the transfer functions in the frequency domain.

Figure 6.3-4: Two linear systems in succession.



Convolution and deconvolution modeling in seismology

$$u(t) = x(t) * g(t) * i(t)$$

Figure 6.3-5: Seismogram as the convolution of the source, structure, and instrument signals.

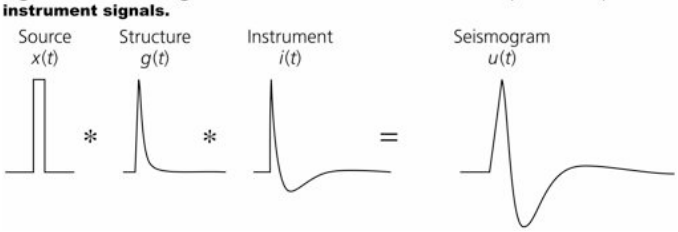
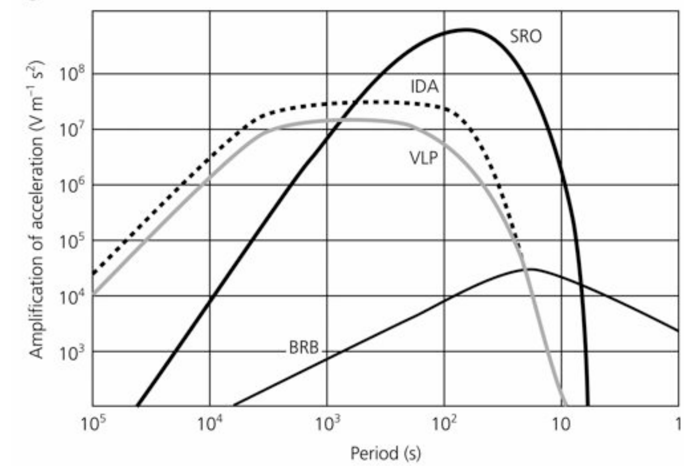
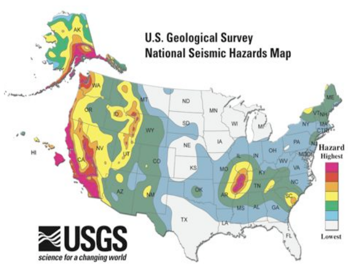
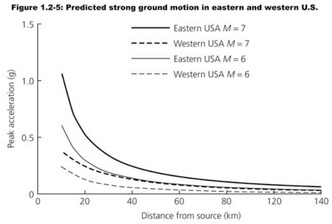


Figure 6.3-6: Transfer functions for various seismometers.



Response of a system in space by convolutions





The Green's function

- The displacement at a point ***x*** and time *t* is

$$u(x,t) = \iint G(x-x';t-t') f(x',t') dt' dV'$$

- Where $G(x-x';t-t')$ is the Green's function, the impulse response to a source at position ***x'*** and time *t'*, and $f(x',t')$ is the distribution of the seismic sources.
- In a general medium

$$u(x,t) = \iint G(x,t;x',t') f(x',t') dt' dV'$$

Inverse filter

- We assume that a seismogram *s*(*t*) results from convolution of a source pulse *w*(*t*), and an earth structure operator *r*(*t*).

$$s(t) = w(t) * r(t) \qquad S(\varpi) = W(\varpi) R(\varpi)$$

- We can create an inverse filter

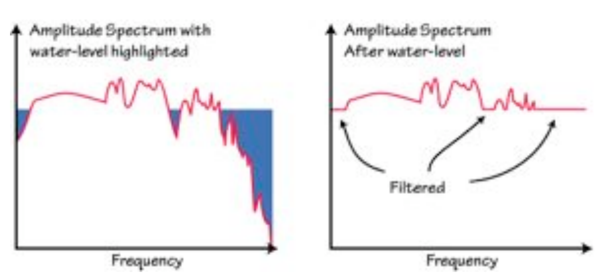
$$w^{-1}(t) * w(t) = \delta(t)$$

- The Fourier transform of the inverse filter is just $1/W(\omega)$, so the deconvolution can be done by dividing the Fourier transforms

$$S(\varpi)/W(\varpi) = R(\varpi)$$

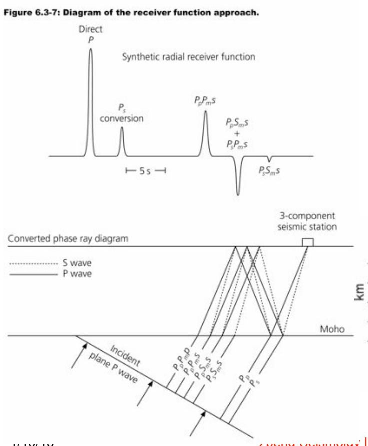
Water-level deconvolution

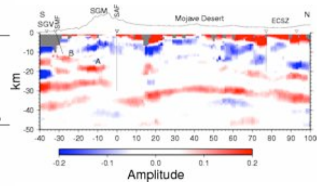
- For $S(\varpi)/W(\varpi) = R(\varpi)$
- What happens if $W(\omega)$ is very small?



http://eqseis.geosc.psu.edu/~cammon/HTML/RftnDocs/seq01.html

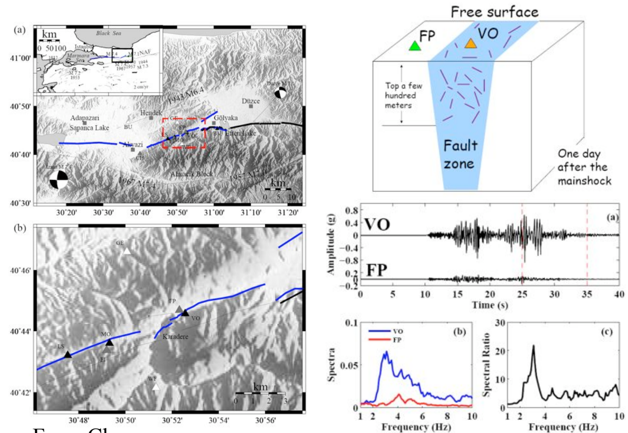
Example of deconvolution





Zhu (EPSL, 2000)

Example of deconvolution



From Chunquan

What we have learned today

- Linear systems
  - Basic models
  - Convolution and deconvolution modeling

Next time

- Finite length signals
- Correlation
- Discreet time series and transforms

Reading: Stein and Wysession Chap. 6.4

Finite length signals

- Consider a window function $b(t)$. Its effect on the data $f(t)$ is represented by multiplying $f(t)$ by $b(t)$.

$$G(\varpi) = \int_{-\infty}^{\infty} b(t) f(t) e^{-i\varpi t} dt$$

$$G(\varpi) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} B(\varpi') e^{i\varpi' t} d\varpi'\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\varpi'') e^{i\varpi'' t} d\varpi''\right] e^{-i\varpi t} dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\varpi')\left[\int_{-\infty}^{\infty} F(\varpi'')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\varpi t + i\varpi' t + i\varpi'' t} dt\right] d\varpi''\right] d\varpi'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\varpi')\left[\int_{-\infty}^{\infty} F(\varpi'')\delta(\varpi - \varpi' - \varpi'') d\varpi''\right] d\varpi'$$

Finite length signals

- Using the sifting properties, we obtain

$$G(\varpi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\varpi')\left[\int_{-\infty}^{\infty} F(\varpi'')\delta(\varpi - \varpi' - \varpi'') d\varpi''\right] d\varpi'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\varpi') F(\varpi - \varpi') d\varpi' = \frac{1}{2\pi} B(\varpi) * F(\varpi)$$

- Hence, the effect of multiplying a time series by a window function is that the spectrum of the time series is convolved with the spectrum of the window function.
- This is what is expected!

Effects of a boxcar window function

$b(t) = 1$ for $-T<t<T$,
$= 0$, otherwise.

Its Fourier transform is:

$$B(\varpi) = \int_{-\infty}^{\infty} e^{-i\varpi t} dt = \frac{e^{-i\varpi t}}{-i\varpi}\Big|_{-T}^{T} = \frac{2\sin\varpi T}{\varpi} = \frac{2T\sin\varpi T}{\varpi T}$$

Figure 6.3-8: A boxcar function in the time and frequency domains.

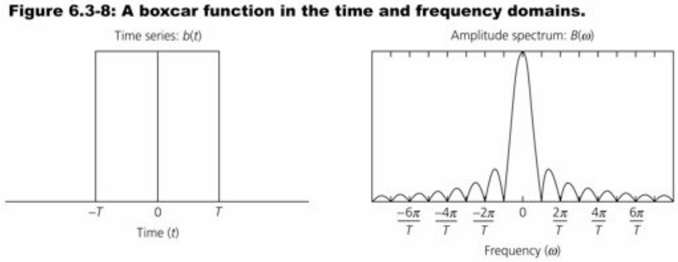
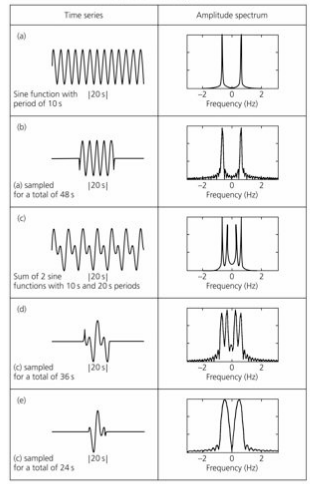


Figure 6.3-9: Effects of windowing time signals on the amplitude spectra.

Data length and frequency resolution



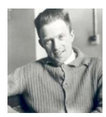
***f(t)* is a sine wave. What's the effect?**

Taking a finite length of record "smears" the delta functions of the infinite length record's spectrum into boader peaks with side lobes.

Input signals contains different frequencies

The frequency resolution, the minimum separation in frequency for which two peaks can be resolved, is proportional to the reciprocal of the total length.

"Uncertainty principle" in time and frequency domains



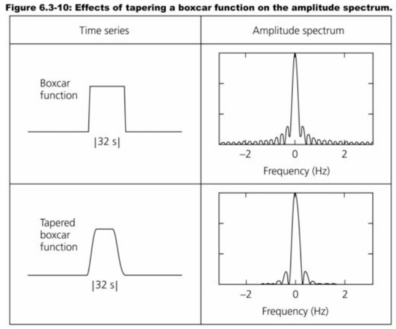
W. Heisenberg

- The product of the "widths" in the two domains is constant.
- For a time domain record with duration T, the resolution in the frequency domain is proportional to 1/T.
- Perfect resolution in frequency requires infinite record length in time.
- Infinite bandwidth in frequency is needed to represent a time function exactly.

Tapered boxcar functions

$$W(t) = \frac{1}{2}\left[1+\cos\frac{\pi(t+T-T_1)}{T_1}\right], for -T<t<-T+T_1$$
$$= \frac{1}{2}\left[1+\cos\frac{\pi(t-T+T_1)}{T_1}\right], for T-T_1<t<T$$

Figure 6.3-10: Effects of tapering a boxcar function on the amplitude spectrum.



The side lobes for the tapered window are reduced, but the central peak is less sharp.

Similarly, band-pass filters are often tapered in the frequency domain.