EAS 8803 - Seismology II

Lec#2: Fourier Transform

Dr. Zhigang Peng, Spring 2008

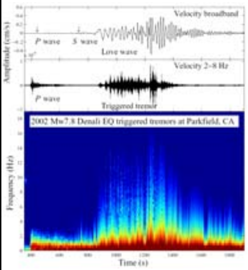
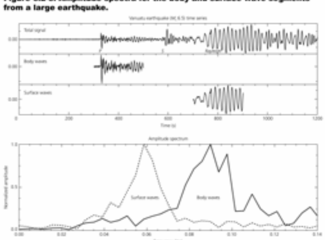


Figure 6.2-3: Amplitude spectra for the body and surface wave segments from a large earthquake.



Last Time

- Course Introduction
  - Class logistics, requirements and policies
  - Class schedule
- Introduction to digital signal processing and its relation to seismological research
- Fourier Series

Reading: Stein and Wysession Chap. 6.1 – 6.2

Today's Outline

- Fourier transforms
- Linear systems

Reading: Stein and Wysession Chap. 6.2-6.3

From Fourier Series to Fourier Transforms

- Fourier Series: a time series expressed in terms of a sum over discrete angular frequencies $\varpi_n = 2n\pi / T$
- Fourier Transforms: a time series expressed as an integral of a continuous range of angular frequencies.
- Fourier Transforms are used in most seismological application, because we regard the waves as continuous functions of angular frequencies

From Fourier Series to Fourier Transforms

- Rewrite $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{i\varpi_n t} = \sum_{n=-\infty}^{\infty} F_n e^{i\varpi_n t} \Delta n$
- where $\Delta n = 1$.
- Since $\Delta\varpi = 2\pi / T \Delta n$
- so that $\Delta n = (T / 2\pi)\Delta\varpi$
- and $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{i\varpi_n t} (T / 2\pi)\Delta\varpi$
- If we let the period T go to infinity,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\varpi) e^{i\varpi t} d\varpi \qquad F(\varpi) = \int_{-\infty}^{\infty} e^{-i\varpi t} f(t) dt$$

Fourier Transforms

- If $f(t)$ is a seismogram that has the dimensions of displacement, its Fourier transform $F(\omega)$ has the dimensions of displacement multiplied by time (from the dt term).
- The Fourier transform can be written in terms of two real-valued functions of ω:

$$F(\varpi) = |F(\varpi)| e^{i\phi(\omega)}$$ Amplitude spectrum

$$|F(\varpi)| = [F(\varpi)F^*(\varpi)]^{1/2} = [\mathrm{Re}^2(F(\varpi)) + \mathrm{Im}^2(F(\varpi))]^{1/2}$$

$$\phi(\varpi) = \tan^{-1}(\mathrm{Im}(F(\varpi)) / \mathrm{Re}(F(\varpi)))$$ Phase spectrum

Figure 6.2-3: Amplitude spectra for the body and surface wave segments from a large earthquake.

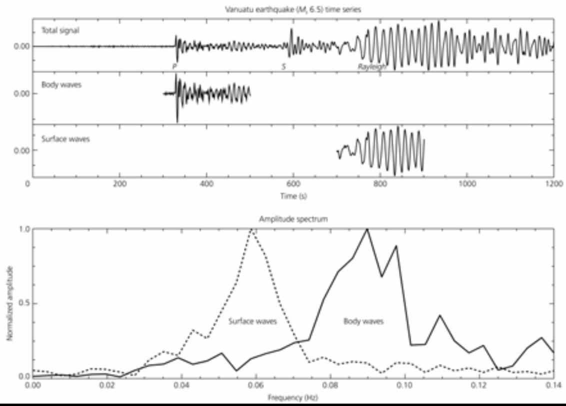
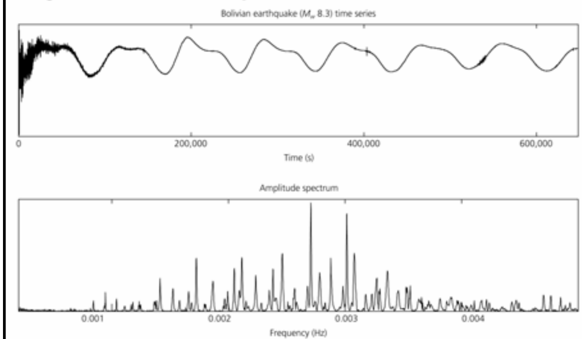


Figure 6.2-4: Amplitude spectra of a vertical-component seismogram from the great 1994 Bolivian earthquake.



Time and frequency domain

- Time domain: time series $f(t)$
- Frequency domain: $F(\omega)$
- Can you think of another pairs of representation in different domain?
- Spatial domain: Distance (d, or wavelength)
- Wavenumber domain: wavenumber (k)

Properties of Fourier Transform (I)

- The Fourier transform is linear: if $F(\omega)$ and $G(\omega)$ are the transforms of $f(t)$ and $g(t)$, then $(aF(\omega) + bG(\omega))$ is the transform of $(af(t) + bg(t))$.
- The Fourier transform of a purely real time function has the symmetry

$$F(-\varpi) = F^*(\varpi)$$

Properties of Fourier Transform (II)

- The Fourier transform of a time series shifted in time is found by changing the phase of the transform: if the transform of $f(t)$ is $F(\omega)$, the transform of $f(t-a)$ is $e^{-i\varpi a}F(\varpi)$
- Similarly, shifting of a Fourier transform in frequency domain causes a phase change in the corresponding time series: the inverse transform of $F(\omega-a)$ is $e^{iat}f(t)$

Properties of Fourier Transform (III)

- The Fourier transform of the derivative of a time function is found by multiplication: $(i\omega)F(\omega)$ is the transform of $df(t)/dt$.
- This makes differentiation easy in the frequency domain, and make it easy to solve differential equations.
- The total energy in a Fourier transform is the same as that in the time series (Parserval theorem):

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\varpi)|^2 d\varpi$$

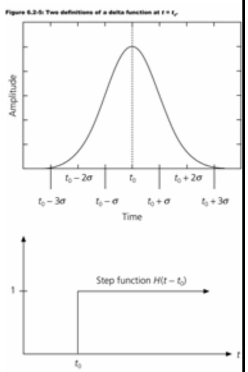
Delta function

Three ways to define it

$$\delta(t-t_0)=\lim_{\sigma\to 0}\frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{-1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right]$$

$$f(t_0)=\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt$$

$$\delta(t-t_0)=dH(t-t_0)/dt$$



Fourier transform of the delta function

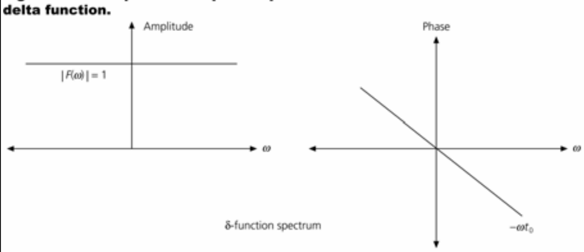
- To find the Fourier transform of the delta function, we use the definition of the transform with $f(t)=\delta(t-t_0)$

$$F(\varpi)=\int_{-\infty}^{\infty} e^{-i\varpi t}\delta(t-t_0)dt=e^{-i\varpi t_0}$$

- The amplitude spectrum is $|F(\varpi)|=\left(e^{-i\varpi t_0}e^{i\varpi t_0}\right)^{1/2}=1$

- The phase spectrum is $\phi(\varpi)=\varpi t_0$

Figure 6.2-6: Amplitude and phase spectra of the Fourier transform of a delta function.



- If the delta function is at time zero,

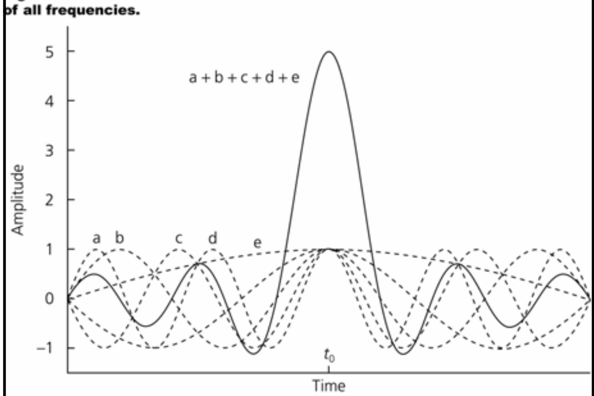
$$F(\varpi)=\int_{-\infty}^{\infty} e^{-i\varpi t}\delta(t)dt=1$$

Fourier transform of the delta function

- The delta function's amplitude spectrum has unit amplitude at all frequencies.
- The output from a linear time-invariant system with delta function input is called impulse response (in time domain), and transfer function (in frequency domain).
- The inverse transform of the delta function

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\varpi t_0}e^{i\varpi t}d\varpi=\delta(t-t_0)$$

Figure 6.2-7: Fourier transform of a delta function as the sum of sinusoids of all frequencies.



Delta function in the frequency domain

- A delta function at angular frequency ϖ_0 has an inverse transform

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\varpi-\varpi_0)e^{i\varpi t}d\varpi=\frac{1}{2\pi}e^{i\varpi_0 t}$$

- So we can express the delta function in terms of its Fourier transform

$$\delta(\varpi-\varpi_0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\varpi_0 t}e^{-i\varpi t}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\varpi_0-\varpi)t}dt$$

Delta function in the frequency domain

- Delta function in angular frequency give the spectra of sinusoids with a single frequency.
- For example, a cosine with frequency ϖ_0

$$f(t) = \cos\varpi_0 t = (e^{i\varpi_0 t} + e^{-i\varpi_0 t})/2$$

- Has a Fourier transform

$$F(\varpi) = \frac{1}{2}\int_{-\infty}^{\infty}(e^{i\varpi_0 t} + e^{-i\varpi_0 t})e^{-i\varpi t}dt = \frac{1}{2}\int_{-\infty}^{\infty}(e^{i(\varpi_0-\varpi)t} + e^{-i(\varpi_0+\varpi)t})dt$$

$$F(\varpi) = \pi[\delta(\varpi - \varpi_0) + \delta(\varpi + \varpi_0)]$$

Summary

- In some cases, frequency domain representation of a time series is simpler.
- So it is common to work in the frequency domain first, and then use the inverse transform to generate the final time series.

Today's Outline

- Fourier transforms

Reading: Stein and Wysession Chap. 6.2-6.3

Next time

- Linear systems

Reading: Stein and Wysession Chap. 6.3